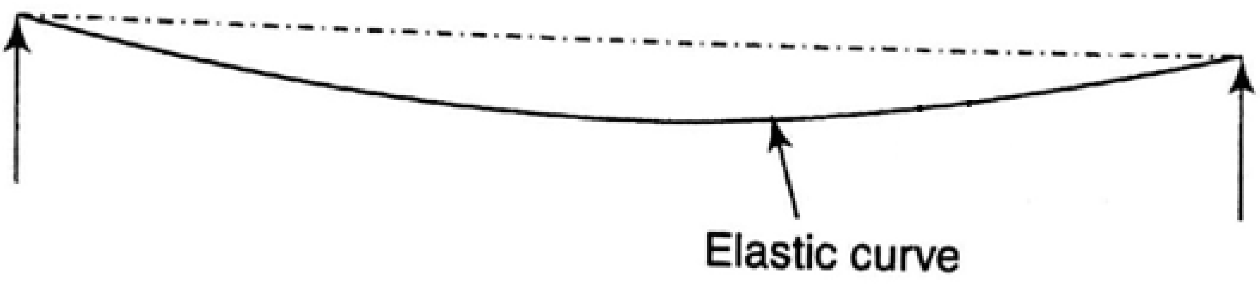
Module 5

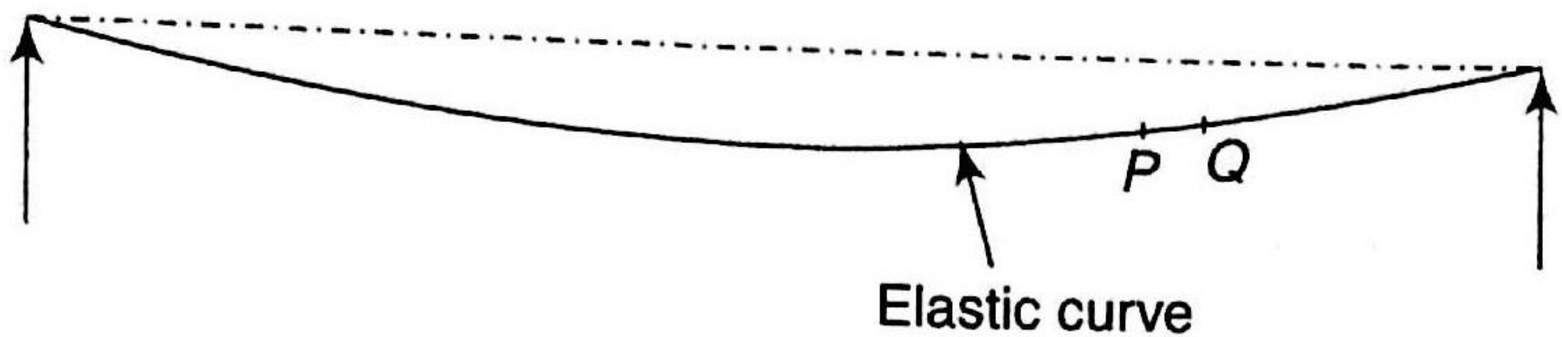
1. Deflection of Beams
2. Transformation of Stress and strain

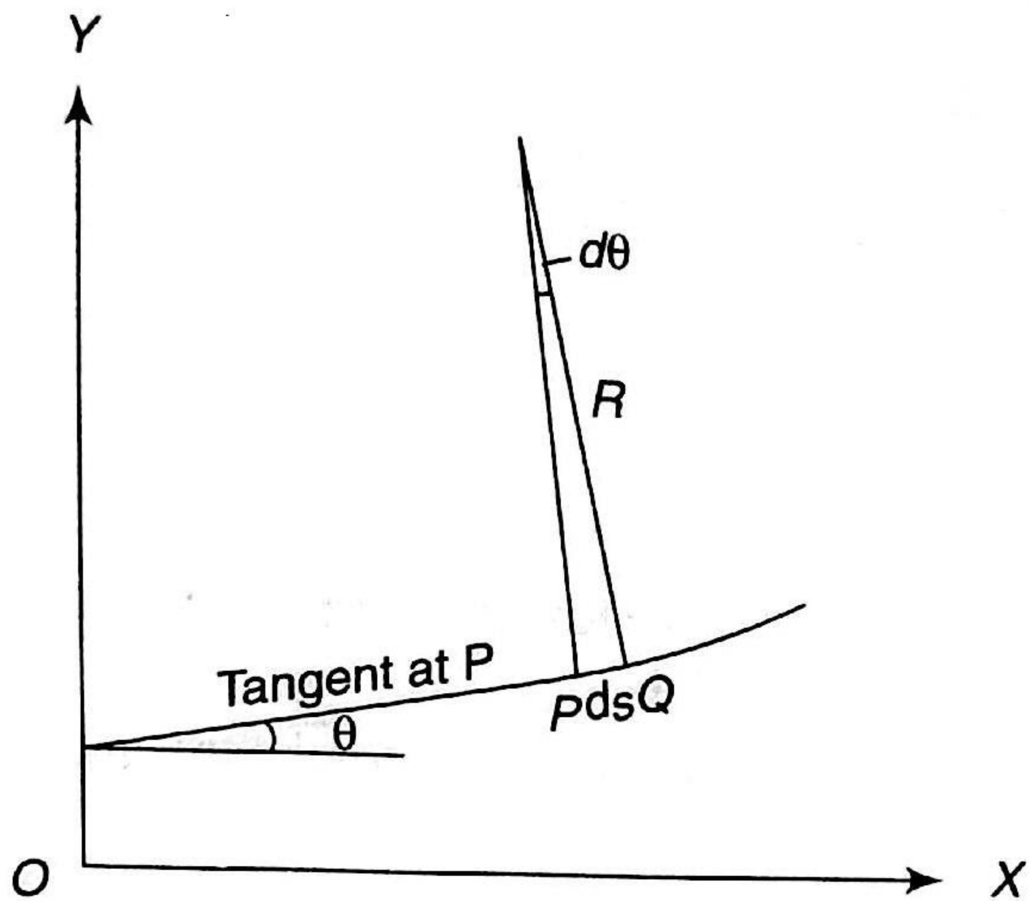
Deflection and Slope of Beams

- As load is applied on a beam, it deflects.
- The deflection can be observed and measured directly.
- Strength and stiffness – design criteria for beams
- Strength criteria – SF & BM
- Stiffness criteria – deflection
- Elastic curve.

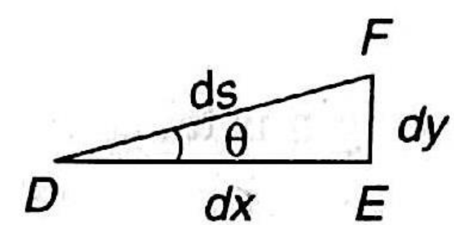

Elastic curve

Beam Differential Equation
OR
Differential Equation for Deflection





Y
dθ
R
Tangent at P
θ
PdsQ
O
X



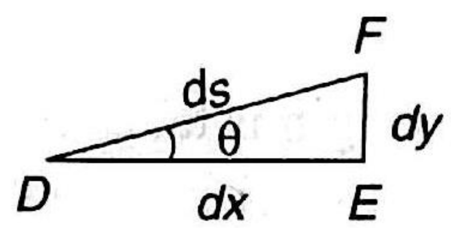
F
ds
dy
θ
D
dx
E

Consider a segment PQ of infinitesimal length ds of the elastic curve of a beam

R be the radius of curvature and $d\theta$ the included angle of the segment

Then, the length $ds = R \cdot d\theta$

As ds is an infinitesimal length, it can be assumed to be the hypotenuse of a right-angled triangle DEF



The slope of the curve at the point P

$$\tan \theta = \frac{dy}{dx} \qquad \text{(i)}$$

Differentiating (i) with respect to x,

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\text{or } \sec^2\theta . \frac{ds}{R} \cdot \frac{1}{dx} = \frac{d^2y}{dx^2}$$

$$\text{or } \frac{\sec^3\theta}{R} = \frac{d^2y}{dx^2} \quad \ldots \left(\because \frac{ds}{dx} = \sec\theta\right)$$

$$\frac{d^2y}{dx^2} = \frac{(1+\tan^2\theta)^{3/2}}{R}$$

$[\sec\theta = (1 + (\tan^2\theta)^{1/2}]$

$$\frac{d^2 y}{dx^2} = \frac{1}{R} = \frac{M}{EI}$$

$$EI \frac{d^2 y}{dx^2} = M$$

the *governing differential equation* of the beam

- Flexural Rigidity
- The moment sustained by an element of the beam is proportional to EI
- Thus EI is an index of the bending (flexural) strength of an element – called *Flexural Rigidity* of the element.

- <u>Some important equations</u>

We have deflection = y

$$\text{Slope} = \frac{dy}{dx}$$

$$\text{Moment, } M = EI\frac{d^2y}{dx^2}$$

$$\text{Shear force, } F = \frac{dM}{dx} = EI\frac{d^3y}{dx^3}$$

$$\text{Load intensity, } w = \frac{dF}{dx} = EI\frac{d^4y}{dx^4}$$

Slope and Deflection at a point

Methods of Solution

1. Double integration method
2. Macaulay’s method
3. Moment area method
4. Conjugate beam method

Double Integration Method

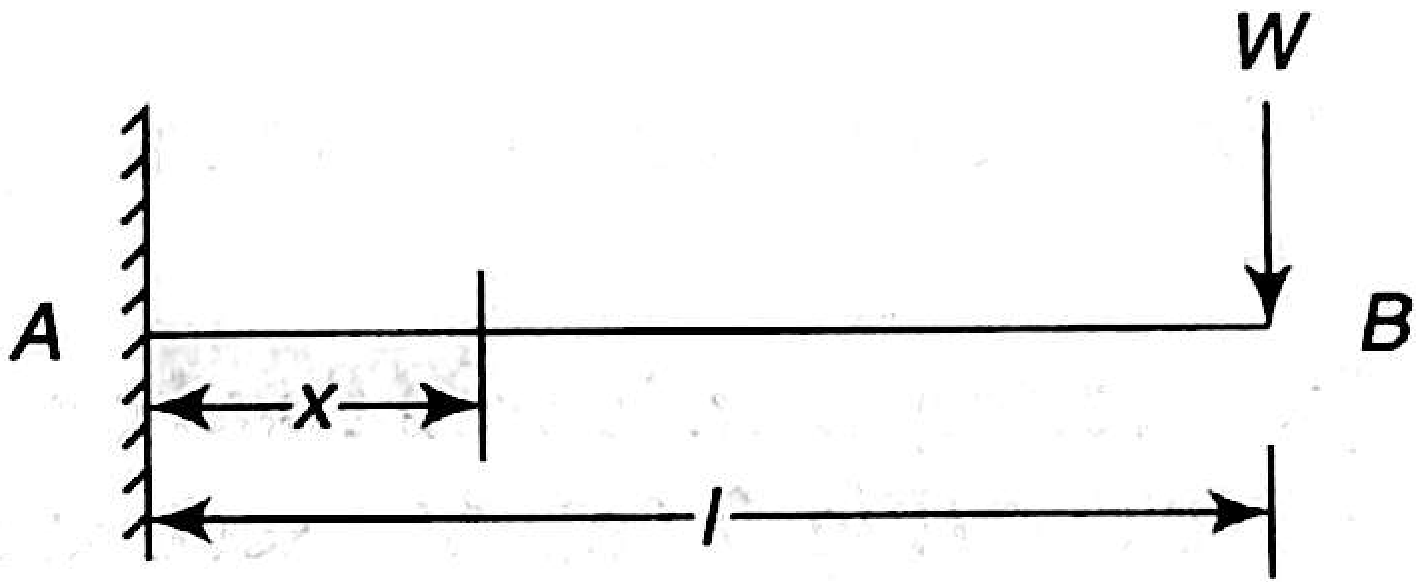
- The beam differential equation is integrated twice – deflection of beam at any c/s.

$$EI\frac{dy}{dx} = \int M \cdot dx + C_1 \quad \text{from which slope can be calculated}$$

$$EI \cdot y = \iint (M \cdot dx) + C_1 x + C_2 \text{ from which deflection is known}$$

- The constants of integration are found by applying the end conditions.

- a) <u>Cantilever with concentrated load at free end</u>



Bending Moment at the section $= -W(l-x)$, being hogging

Or $EI\frac{d^2y}{dx^2} = -W(l-x)$

Integrating, $EI\frac{dy}{dx} = -W\left(lx - \frac{x^2}{2}\right) + C_1$

At $x = 0, \frac{dy}{dx} = 0$, therefore $C_1 = 0$,

Thus $EI\frac{dy}{dx} = -W\left(lx - \frac{x2}{2}\right)$

Or $\boldsymbol{Slope,\ \frac{dy}{dx} = -\frac{W}{2EI}(2lx - x^2)}$

$$\text{EI}\,\frac{dy}{dx} = -W\left(lx - \frac{x2}{2}\right)$$

Integrating again, EI y = $-W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2$

At x = 0, y = 0, therefore $C_2 = 0$,

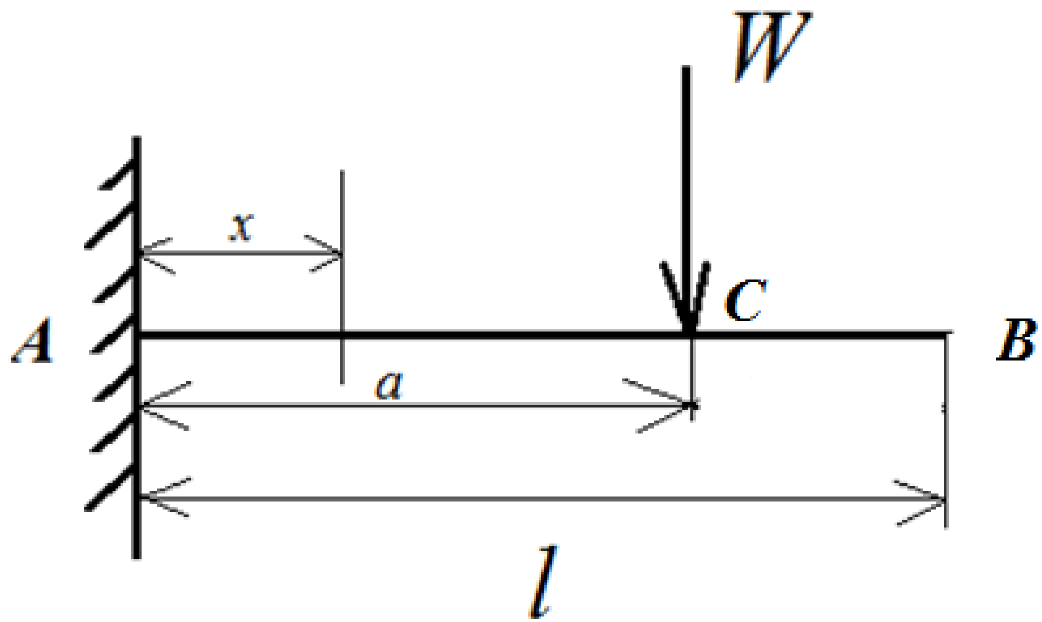
Thus EIy = -W $\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$

Or ***Deflection, y =*** $-\frac{W}{6EI}\left(3lx^2 - x^3\right)$

At the free end, x = l, the slope and deflection are maximum and are given by

$$\text{Slope} = -\frac{Wl^2}{2EI} \text{ and deflection} = -\frac{Wl^3}{3EI}$$

b) Concentrated load not at free end



- Between A and C at any distance x from A,
  M = -W (a-x)
- Equations of slope and deflection can be obtained as in previous case (replacing ℓ by a)

$$\text{Slope,} \quad \frac{dy}{dx} = -\frac{W}{2EI}\left(2ax - x^2\right)$$

$$\text{Deflection,}\ y = -\frac{W}{6EI}\left(3ax^2 - x^3\right)$$

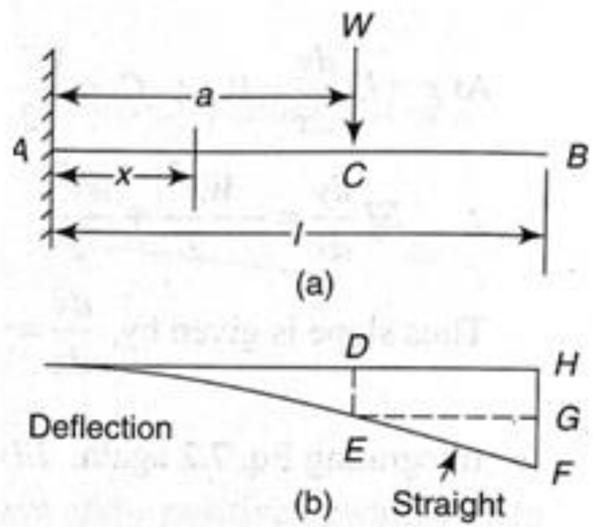
At C, x=a; hence $\frac{dy}{dx} = -\frac{Wa^2}{2EI}$

and $y = -\frac{Wa^3}{3EI}$

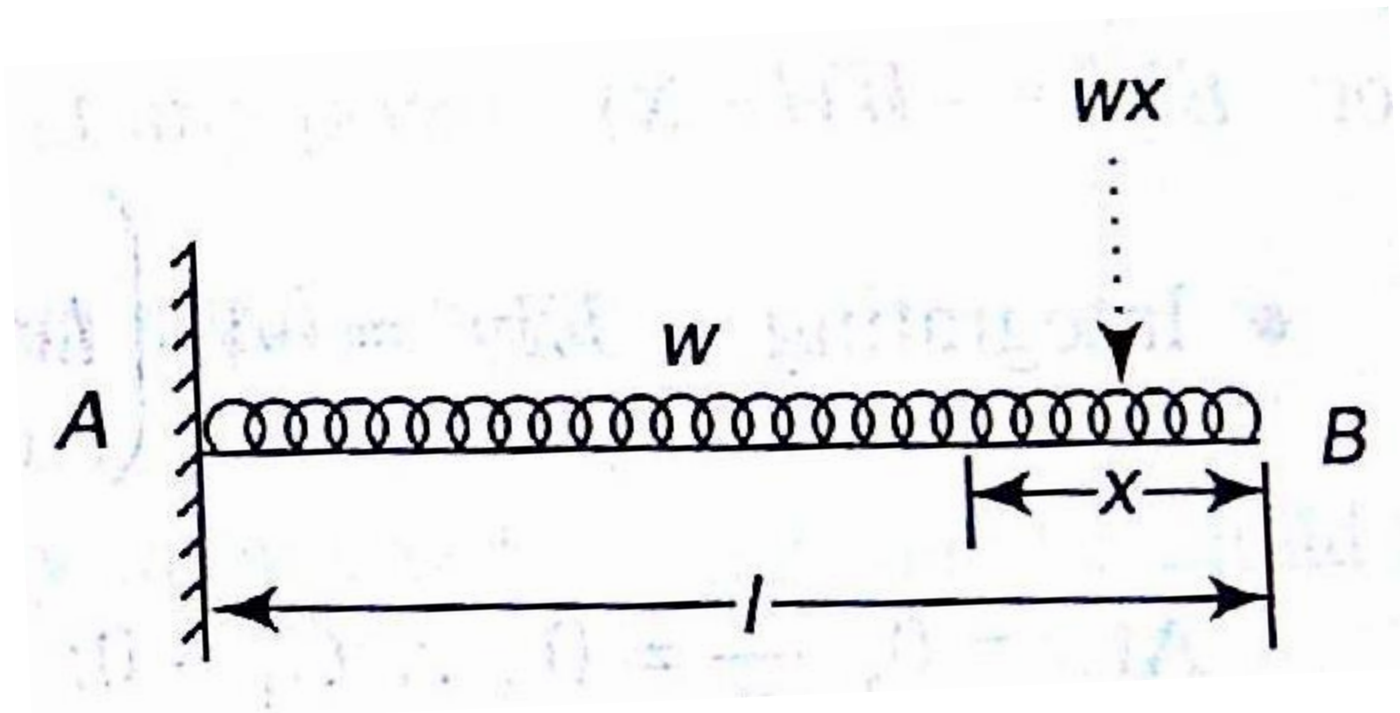
- The beam will bend only between A and C and between B and C it will remain straight (as BM between B and C = 0)
- Hence slope at B = slope at C = dy/dx = GF/GE = $-\frac{Wa^2}{2EI}$
- Now deflection at B = deflection at C + GF
- = deflection at C +($-\frac{Wa^2}{2EI}$)GE

ie, Deflection at B = $-\frac{Wa^3}{3EI}-\frac{Wa^2}{2EI}(l\text{-}a)$

If W is at the midpoint, deflection = $\left[\frac{W(l/2)^3}{3EI}+\frac{W(l/2)^2}{2EI}\cdot\frac{l}{2}\right]=\frac{5Wl^3}{48EI}$



c) UDL on whole span



At a section at a distance x from the free end,

$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}$$

Integrating, $EI\dfrac{dy}{dx} = -\dfrac{wx^3}{6} + C_1$

At $x = l$, $\dfrac{dy}{dx} = 0$, $\therefore\ C_1 = \dfrac{wl^3}{6}$

Thus, $EI\dfrac{dy}{dx} = -\dfrac{wx^3}{6} + \dfrac{wl^3}{6} = \dfrac{w}{6}(l^3 - x^3)$

Integrating again, $EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x + C_2$

At A, $x = l, y = 0$, $\therefore\ 0 = -\frac{wl^4}{24} + \frac{wl^3}{6} \cdot l + C_2$

or $C_2 = -\frac{wl^4}{8}$

Thus, $$EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x - \frac{wl^4}{8}$$

Therefore, slope and deflection are given by,

$$\frac{dy}{dx} = \frac{w}{6EI}(l^3 - x^3) \text{ and } y = -\frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$$

$$\text{Maximum slope} = \frac{wl^3}{6EI} \text{ at } x = 0$$

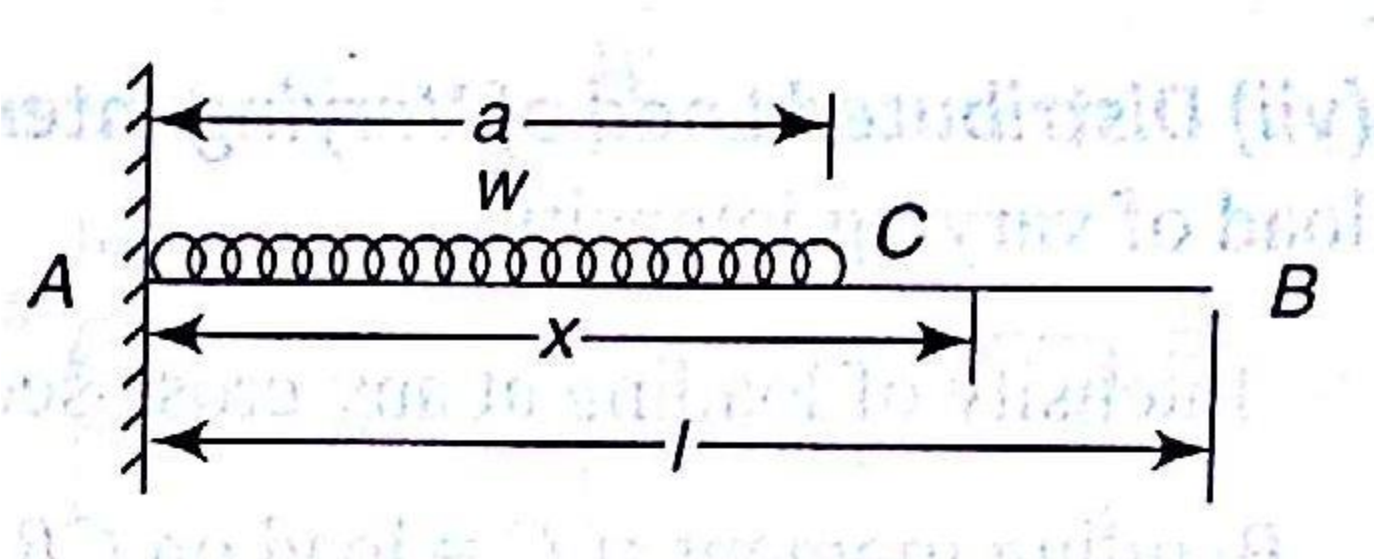
$$\text{Maximum deflection} = -\frac{wl^4}{8EI} \text{ at } x = 0$$

If origin is taken at the fixed end, slope and deflection can be worked out to be

$$y' = -\frac{w}{6EI}(3l^2x - 3lx^2 + x^3); \quad y = -\frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4)$$

d) UDL on a part of span from fixed end

- Homework



At C, $\dfrac{dy}{dx} = -\dfrac{wa^3}{6EI}$ and $y_c = -\dfrac{wa^4}{8EI} \cdots (l = x = a)$

Between CB, at any section at a distance x from A, $M = 0$,

$$\therefore \quad EI\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{dy}{dx} = C_1$$

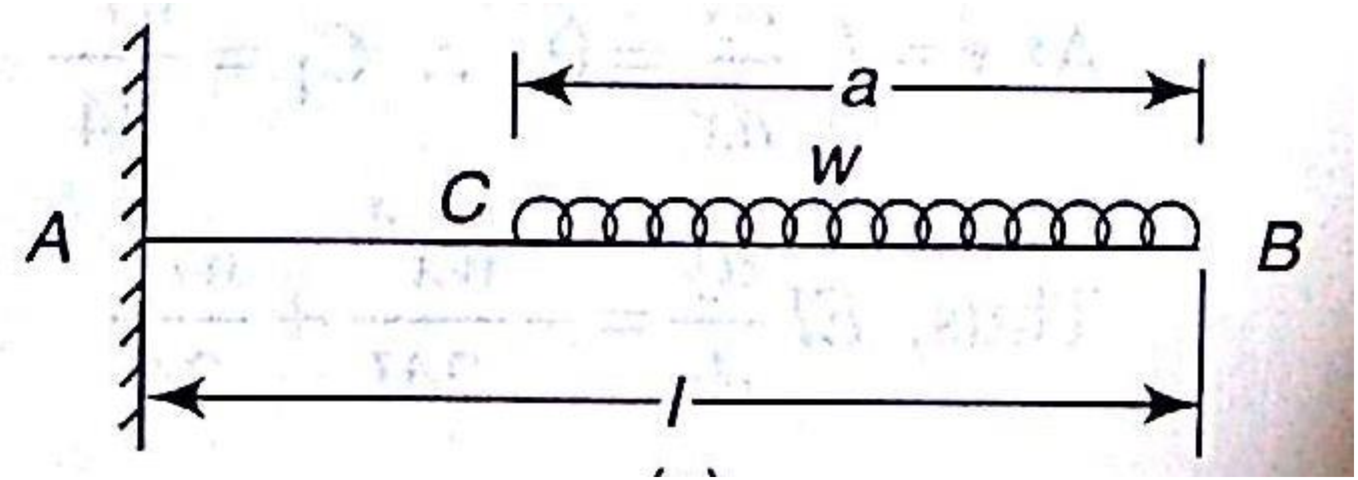
i.e. the slope is constant between CB and is equal to slope at C.

$$\frac{dy}{dx} = y' = \frac{GF}{GE} = -\frac{wa^3}{6EI} \quad \text{or} \quad GF = y' \cdot GE$$

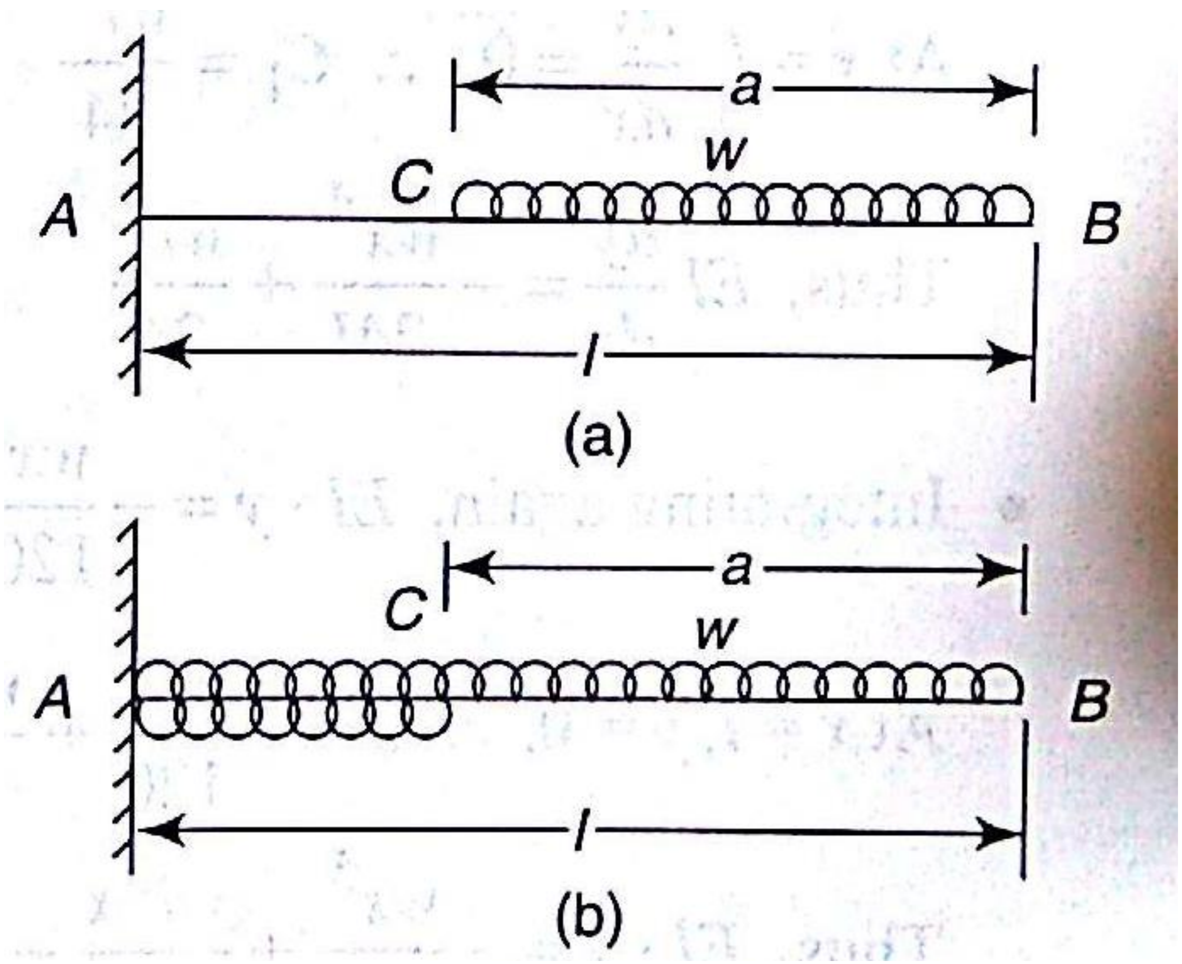
$$\begin{aligned}\text{Deflection at } B &= \text{Deflection at } C + GF \\ &= \text{Deflection at } C + y' \cdot GE \\ &= -\frac{wa^4}{8EI} - \frac{wa^3}{6EI} \cdot (l - a)\end{aligned}$$

e) UDL on a part of span from free end

- Homework



(v) *Uniformly Distributed Load on a Part of Span from Free End* The slope and the deflection at B can be found by first considering the cantilever loaded for the whole span (Fig. 7.8a) and then deducting the effect for the span loaded from A to C upwards (Fig. 7.8b).



Thus slope, $\frac{dy}{dx} = \frac{wl^3}{6EI} - \frac{w(l-a)^3}{6EI}$

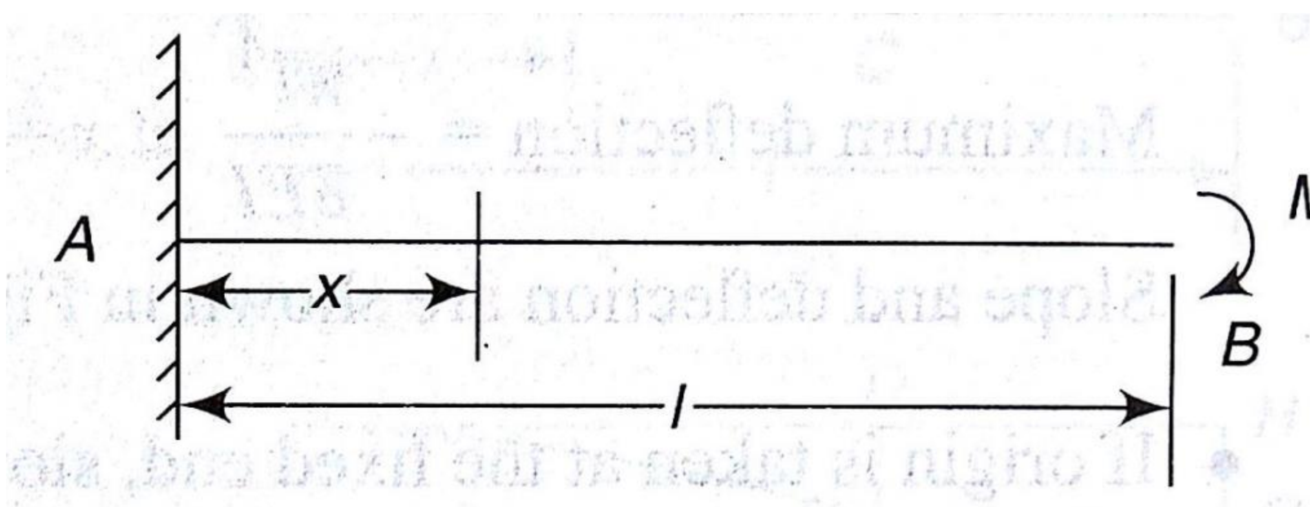
Deflection can be found as follows,

- For whole span having uniformly distributed load, $y_b = \frac{wl^4}{8EI}$ (downwards)
- For span loaded between AC,

$$\frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI} \cdot a \quad \text{(upwards)}$$

Thus deflection of B (downwards) $= \frac{wl^4}{8EI} - \left[\frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI} \cdot a \right]$

f) A couple at the free end



$$EI\frac{d^2y}{dx^2} = -M$$

Integrating, $EI\frac{dy}{dx} = -Mx + C_1$

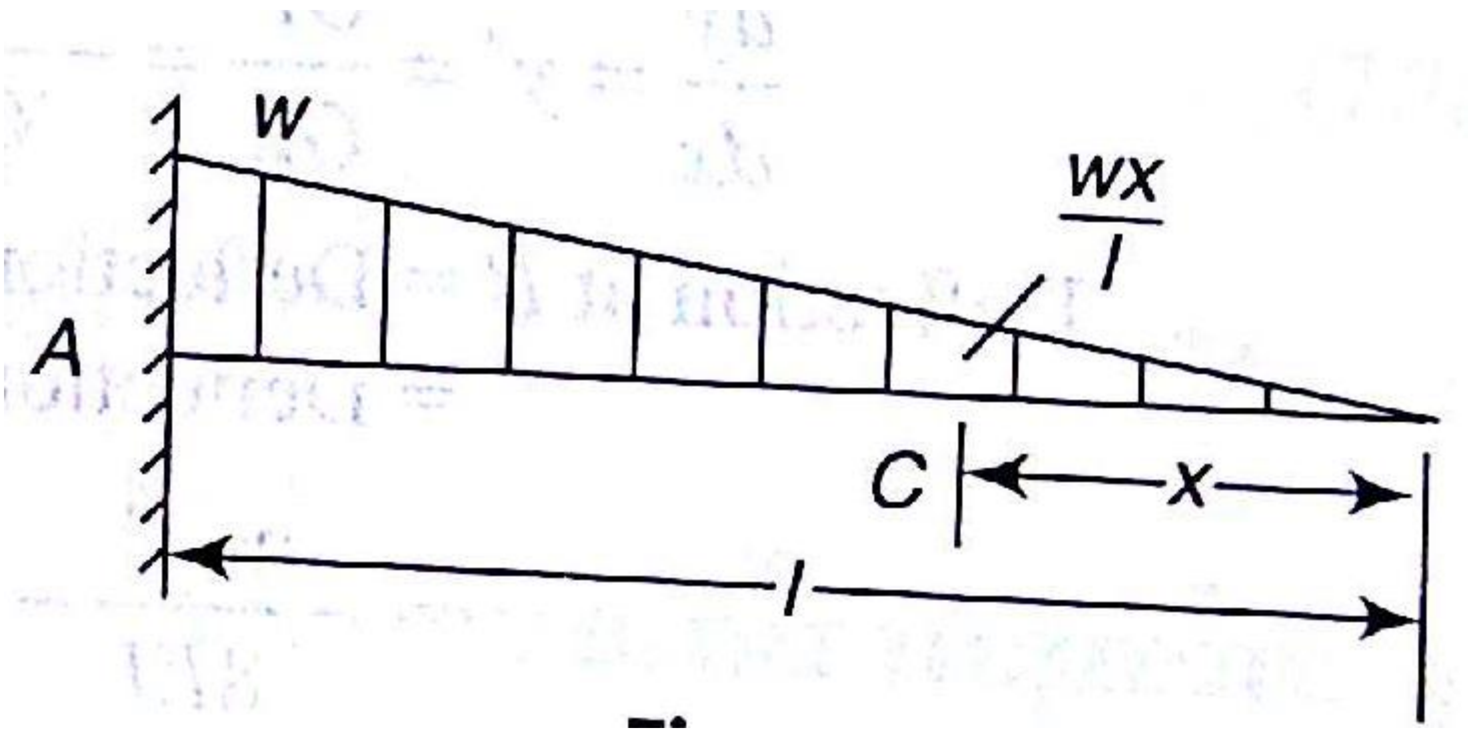
At $x = 0$, $\frac{dy}{dx} = 0$, $\therefore$ $C_1 = 0$; Thus, $EI\frac{dy}{dx} = -Mx$

Integrating again, $EI \cdot y = -M\frac{x^2}{2} + C_2$

At $x = 0$, $y = 0$, $\therefore$ $C_2 = 0$; Thus, $EIy = -\frac{M}{2}x^2$

$\frac{dy}{dx} = -\frac{M}{EI}x$ (linear) and $y = -\frac{M}{2EI}x^2$ (parabola)

g) Distributed load of varying intensity, zero at free end



Intensity of loading at any cross-section C at a distance x from free end $= \dfrac{wx}{l}$

Bending moment at C = load on CB X distance of centre of load

$$= \left(\frac{1}{2}\frac{wx}{l}\cdot x\right)\cdot\frac{x}{3} = \frac{wx^3}{6l}$$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6l}$$

Integrating, $EI\dfrac{dy}{dx} = -\dfrac{wx^4}{24l} + C_1$

At $x = l$, $\dfrac{dy}{dx} = 0$, $\therefore\ C_1 = \dfrac{wl^3}{24}$

Thus, $EI\dfrac{dy}{dx} = -\dfrac{wx^4}{24l} + \dfrac{wl^3}{24}$

Integrating again, $EI \cdot y = -\dfrac{wx^5}{120l} + \dfrac{wl^3 x}{24} + C_2$

At $x = l, y = 0, \therefore\ 0 = -\dfrac{wl^4}{120} + \dfrac{wl^4}{24} + C_2$ or $C_2 = -\dfrac{wl^4}{30}$

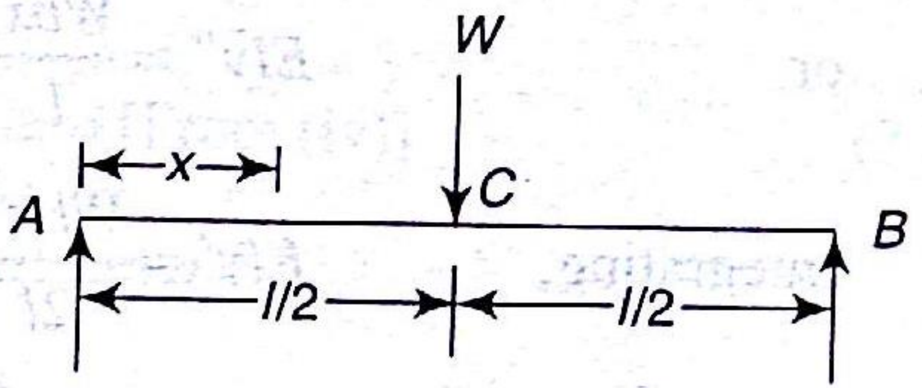
Thus, $EI \cdot y = -\dfrac{wx^5}{120l} + \dfrac{wl^3 x}{24} - \dfrac{wl^4}{30}$

Therefore, slope and deflection at free end i.e. at $x = 0$,

$$\frac{dy}{dx} = \frac{wl^3}{24EI} \text{ and } y = -\frac{wl^4}{30EI}$$

Simply supported Beams

a) Concentrated load at midspan



$\therefore \quad R_a = R_b = W/2$

Consider a section from A (origin at A),

$$M = \frac{W}{2}x$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = \frac{W}{2}x$$

Integrating, $EI\dfrac{dy}{dx} = \dfrac{Wx^2}{4} + C_1$

At $x = \dfrac{l}{2}$, $\dfrac{dy}{dx} = 0$, $\therefore$ $C_1 = -\dfrac{Wl^2}{16}$ $\therefore$ $EI\dfrac{dy}{dx} = \dfrac{Wx^2}{4} - \dfrac{Wl^2}{16}$

Integrating again, $EIy = \dfrac{Wx^3}{12} - \dfrac{Wl^2x}{16} + C_2$

At $x = 0, y = 0$, $\therefore$ $EIy = \dfrac{Wx^3}{12} - \dfrac{Wl^2x}{16}$

Therefore, slope and deflection are given by,

$$\frac{dy}{dx} = -\frac{W}{16EI}(l^2 - 4x^2) \text{ and } y = -\frac{W}{48EI}(3l^2x - 4x^3)$$

At A, $x = 0$, $\therefore$ slope $= -\dfrac{Wl^2}{16EI}$

Deflection at $C = -\dfrac{W}{48EI}\left(3l^2 \cdot \dfrac{l}{2} - 4 \cdot \dfrac{l^3}{8}\right) = -\dfrac{Wl^3}{48EI}$

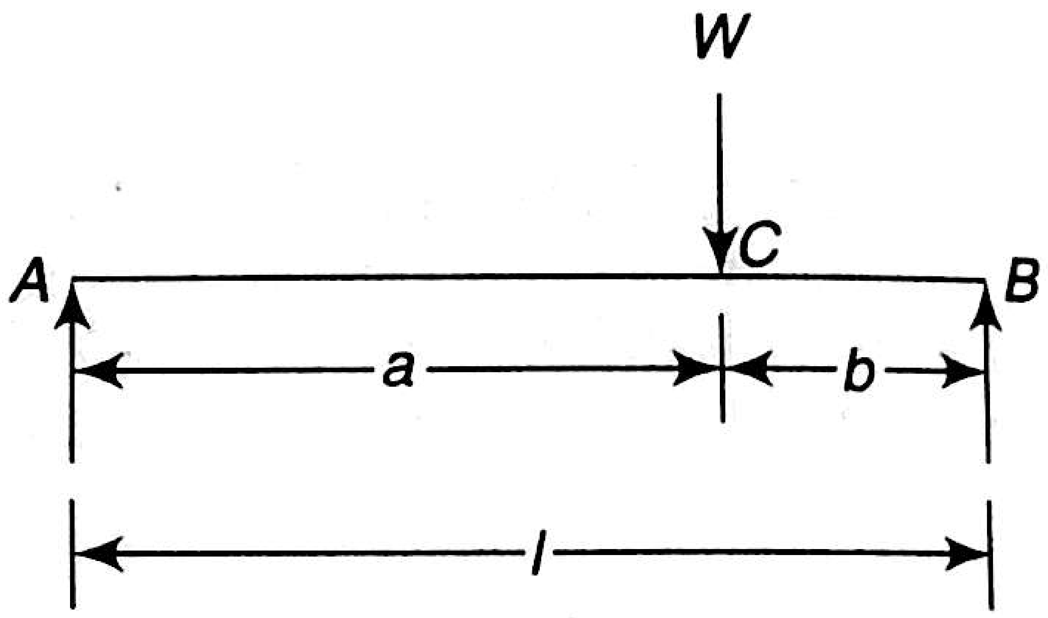
Slope and deflection for the portion *CB* is symmetric as for *AC*. However, equations for the portion *CB* with *A* as origin can also be formed in the following form:

$$\frac{dy}{dx} = -\frac{W}{16EI}(4x^2 - 8lx + 3l^2)$$

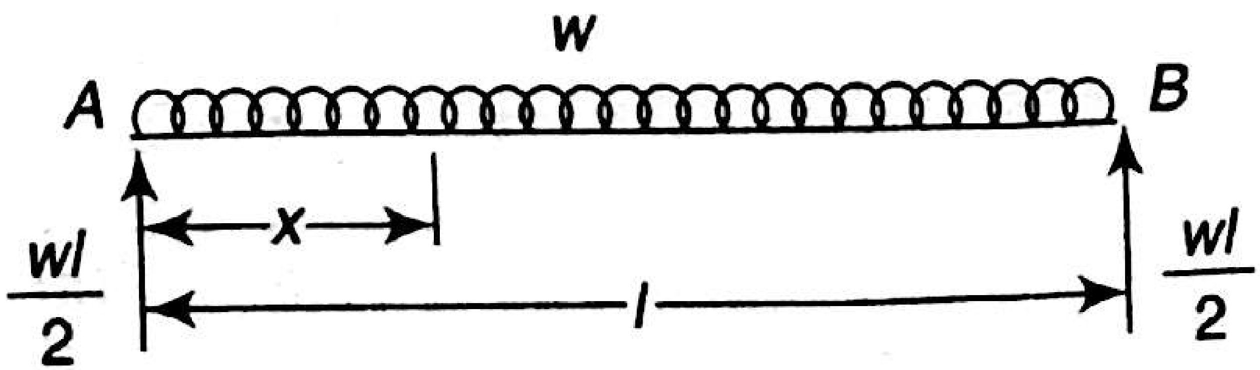
$$y = -\frac{W}{48EI}(4x^3 + 9l^2x - l^3 - 12lx^2)$$

b) Eccentric concentrated load

- Homework



c) UDL on whole span



$\therefore \quad R_a = R_b = wl/2$

Consider a section of the beam from A (origin at A),

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}$$

Integrating, $EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1$

At $x = \frac{l}{2}, \frac{dy}{dx} = 0$, $\therefore\ 0 = \frac{wl}{4} \cdot \frac{l^2}{4} - \frac{w}{6} \cdot \frac{l^3}{8} + C_1$ or $C_1 = -\frac{wl^3}{24}$

$$\therefore \quad EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}$$

Integrating again, $EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24}x + C_2$

At $x = 0, y = 0$, $\therefore\ C_2 = 0$ $\therefore\ EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24}x$

The maximum deflection is at the midspan, i.e., at $x = l/2$,

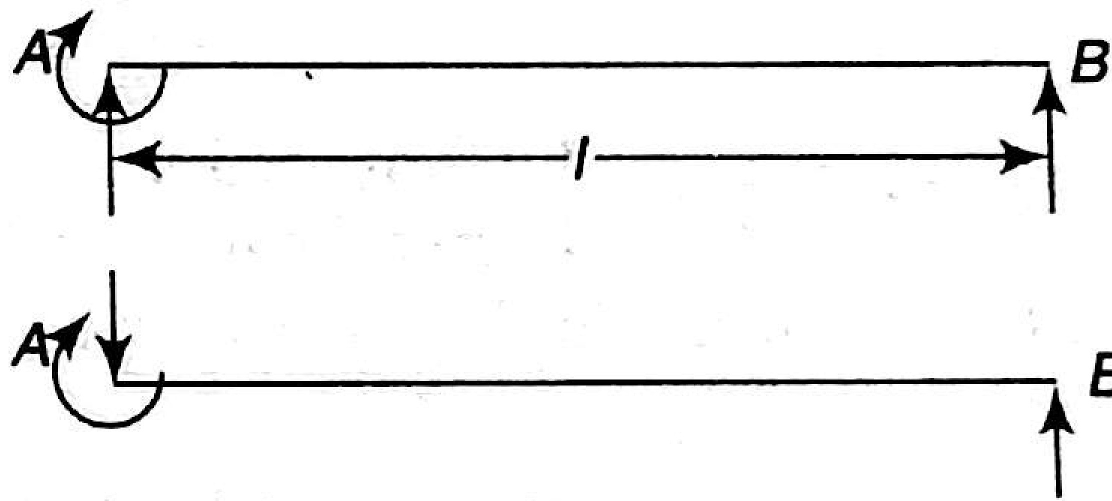
$$y_{\max} = \frac{1}{EI}\left[\frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^3}{24EI}\left(\frac{l}{2}\right)\right] = -\frac{5}{384EI}wl^4$$

Slope at A, $(x = 0)$, $EI\frac{dy}{dx} = -\frac{wl^3}{24}$ or $\frac{dy}{dx} = -\frac{wl^3}{24EI}$

d) Distributed load of varying intensity

- Homework

e) Couple at one end



Taking moments about A, $R_b \times l = M$ or $R_b = \dfrac{M}{l}$ (↑)

Similarly, $R_a = \dfrac{M}{l}$ (↓)

At any section x from A, $EI \dfrac{d^2y}{dx^2} = -\dfrac{M}{l}x + M$

Integrating, $$EI\frac{dy}{dx} = -\frac{Mx^2}{2l} + Mx + C_1$$

Integrating again, $EIy = -\dfrac{Mx^3}{6l} + \dfrac{M}{2}x^2 + C_1x + C_2$

- At A, $x = 0$, $y = 0$

$$EIy = -\frac{Mx^3}{6l} + \frac{M}{2}x^2 + C_1x + C_2 \text{ or } C_2 = 0$$

At B, $x = l$, $y = 0$

or $$0 = -\frac{Mx^3}{6l} + \frac{M}{2}x^2 + C_1x$$

or $$C_1 = \frac{Ml^2}{6l} - \frac{M}{2l}l^2 = -\frac{Ml}{3}$$

Thus slope and deflection equations are

$$EI\frac{dy}{dx} = -\frac{Mx^2}{2l} + Mx - \frac{Ml}{3} = -\frac{M}{6l}(3x^2 - 6lx + 2l^2)$$

and

$$EIy = -\frac{Mx^3}{6l} + \frac{M}{2}x^2 + C_1x = -\frac{M}{6l}(x^3 - 3lx^2 + 2l^2x)$$

$$\text{Slope at } A = -\frac{M}{6lEI}(2l^2) = -\frac{Ml}{3EI}$$

$$\text{Slope at } B = -\frac{M}{6lEI}(3l^2 - 6l^2 + 2l^2) = \frac{Ml}{6EI}$$

Maximum deflection will be where slope is zero, i.e.,

$$3x^2 - 6lx + 2l^2 = 0 \text{ or } x = 0.423\ l$$

Thus maximum deflection,

$$y_{\max} = -\frac{M}{6EIl}(x^3 - 3lx^2 + 2l^2 x)$$

$$= -\frac{M}{6EIl}\left[(0.423l)^3 - 3l \times (0.423l)^2 + 2l^2 \times 0.423\right]$$

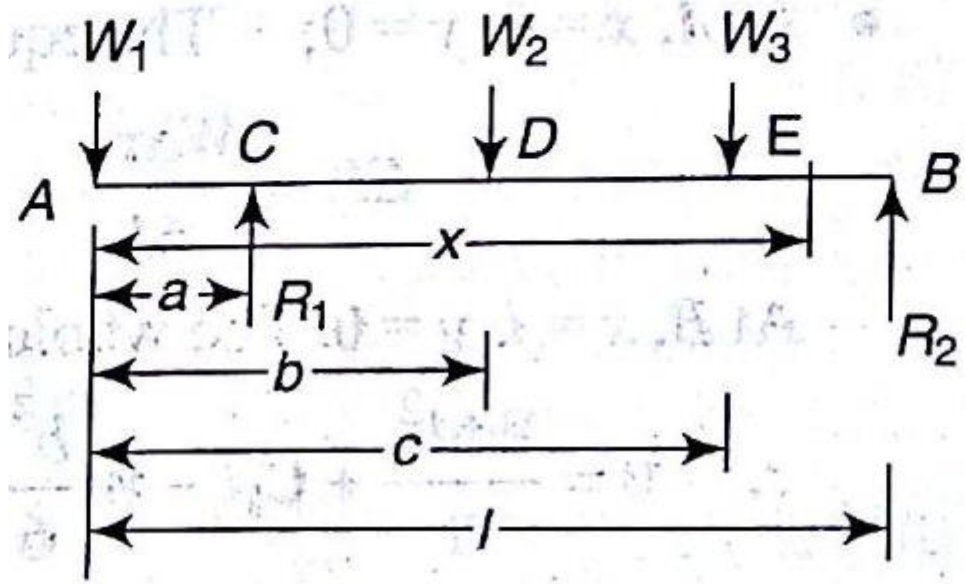
$$= -\frac{0.64Ml^2}{6EI}$$

Macaulay’s method
OR
Method of Singularity function

While applying the double integration method, a separate expression for the bending moment is needed to be written for each section of the beam, each producing a different equation with its own constants of integration.

The method is convenient for simple cases

In Macaulay’s method, a single equation is written for the bending moment for all the portions of the beam. The equation is formed in such a way that the same constants of integration are applicable to all portions.



$$EI\frac{d^2y}{dx^2} = M = -W_1x \mid + R_1(x-a) \mid - W_2(x-b) \mid - W_3(x-c)$$

In the above expression, there are separation lines.

- The portion to the left of the first separation line is valid for the portion *AC*.
- The portion to the left of the second separation line is valid for the portion *CD*.
- The portion to the left of the third separation line is valid for the portion *DE*.
- The whole of the expression is valid for the portion *EB*.

It may be noted that the same expression is applicable to all the portions of the beam if all negative terms inside the brackets are omitted for a particular section. If x is less than c, then the last term is omitted. If x is less than b, then the last two terms are omitted and so on. While integrating, the brackets are integrated as a whole, i.e.,

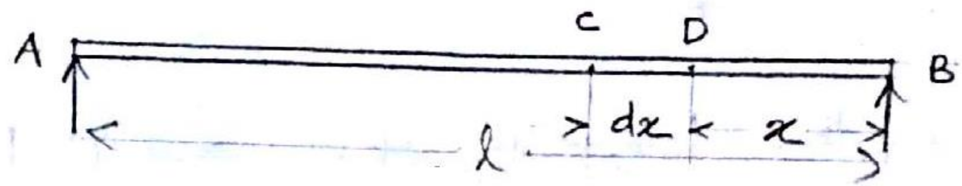
$$EI\frac{d^2y}{dx^2} = M = -W_1x \mid + R_1(x-a) \mid - W_2(x-b) \mid - W_3(x-c)$$

$$EI\frac{dy}{dx} = -W_1\frac{x^2}{2} + C_1 \Big| + \frac{R_1}{2}(x-a)^2 \Big| - \frac{W_2}{2}(x-b)^2 \Big| - \frac{W_3}{2}(x-c)^2$$

$$EIy = -W_1\frac{x^3}{6} + C_1x + C_2 \Big| + \frac{R_1}{6}(x-a)^3 \Big| - \frac{W_2}{6}(x-b)^3 \Big| - \frac{W_3}{6}(x-c)^3$$

Moment Area Method
OR
Mohr’s Theorems

- Convenient for beams acted upon with point loads where BMD consists of triangles and rectangles.
- For the case of UDL, Macaulay’s method is most uitable.


A
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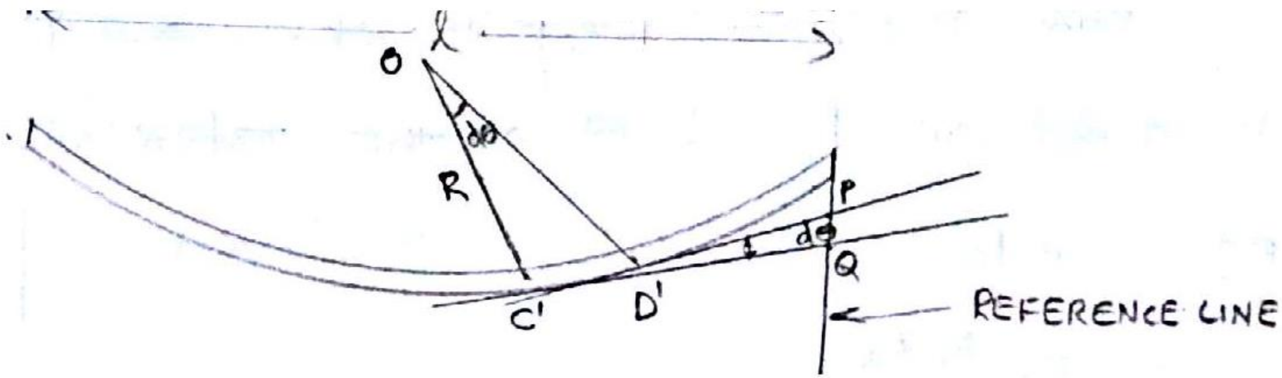
l
O
dθ
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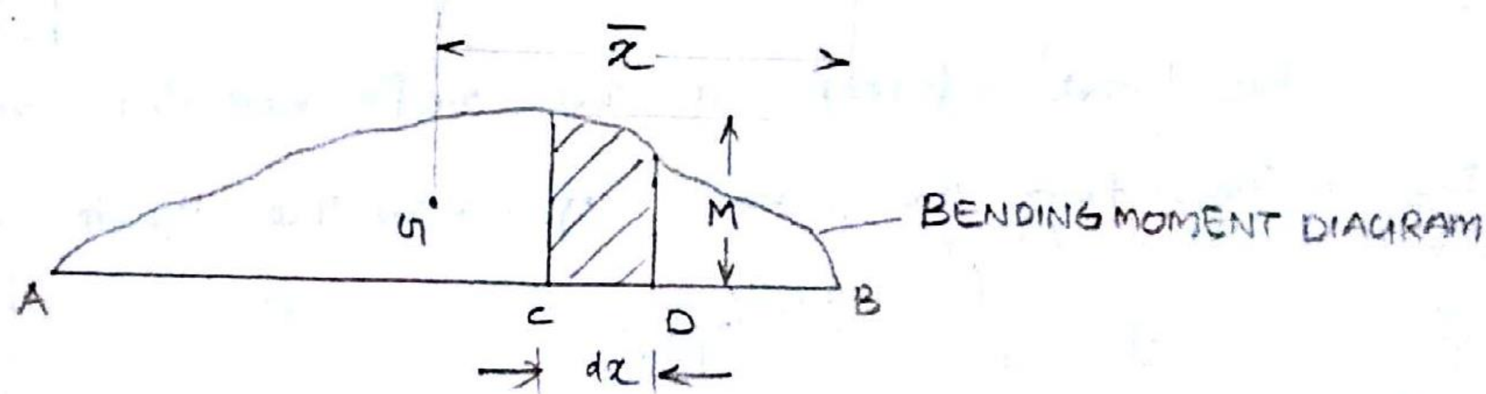
D'
REFERENCE LINE


x̄
G
M
BENDING MOMENT DIAGRAM
A
C
D
B
dx

Now consider an element of small length CD of the beam at a distance x from B as shown in figure. Let M = Bending moment b/w C and D.

dx = length of CD.

R = radius of the bent beam

$d\theta$ = angle included b/w the tangents at C' and D'

or it is the change in slope over the elementary portion 'dx'

A = area of the bending moment diagram over the entire span

$\bar{x}$ = Horizontal distance of the centre of gravity (G) of the entire BM diagram from the reference line.

θ = Angle in radians included b/w the tangents drawn at the two extremities of the beam.

ie, b/w A and B and facing the reference line.

From the figure $dx = Rd\theta$

$$\text{Or} \quad d\theta = \frac{dx}{R} = \frac{M}{EI} dx,$$

This equation gives the change of slope between C and D

∴ The total change of slope from A to B may be found by integrating the above eqn: b/w the limits 0 to l

$$\therefore \theta = \int_0^l \frac{M\,dx}{EI}$$

$$= \frac{1}{EI} \int_0^l M\,dx.$$

But $\int_0^l M\,dx$ = Area of the Bending moment diagram over the entire span $= A$.

$$\therefore \theta = \frac{A}{EI} \longrightarrow (1)$$

- From the above **Mohr’s first moment-area theorem** can be stated as below:
- “The difference of slopes between any two points on an elastic curve of a beam is equal to the net area of the BMD between these points divided by EI”.

Now draw the tangents at C' and D'. Let these two tangents meet at P and Q on the reference line as shown in fig:

Now $PQ = x\,d\theta$

$$= x\,\frac{M\,dx}{EI}.$$

The total intercept on the reference line may be found out by integrating the above eqn: b/w the limits 0 and l.

$$\therefore\ y = \int_0^l \frac{M\,dx \times x}{EI}$$

$$= \frac{1}{EI}\int_0^l Mx\,dx$$

But $Mx\,dx$ = moment of the area of the BM diagram over the portion dx about the reference line.

$\therefore \int_0^l Mx\,dx$ = Moment of the area of the BM diagram over the entire span 'l' about the reference line.

$$= A\bar{x}.$$

$$\therefore\ y = \frac{A\bar{x}}{EI} \longrightarrow (2)$$

The results given by eqns (1) and (2) are known as Mohr's theorem.

- The above equation leads to the statement of Mohr’s second theorem.
- “The intercepts on a given line between the tangents to the elastic curve of a beam at any two points is equal to the net moment taken about the line of the area of the BMD between the two points divided by EI”.

Case I : Cantilever Beams

1. Point load at the free end.

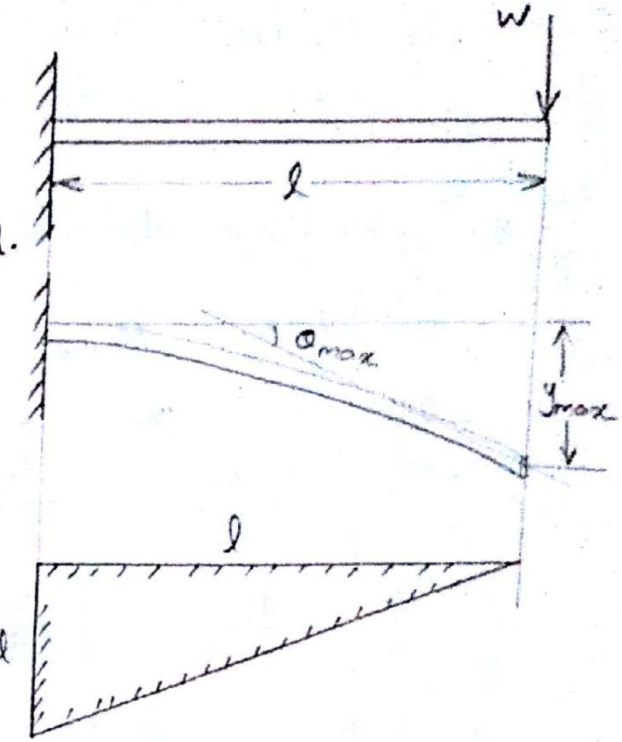


Fig shows a cantilever with a concentrated load W acting at the free end. The slope and deflection will be max. at the free end.

$$\theta_{max} = \frac{A}{EI}$$

where $A = \frac{1}{2} l \times Wl$

$$\therefore \theta_{max} = \frac{Wl^2}{2EI}$$

Now
$$y_{max} = \frac{A\bar{x}}{EI}$$

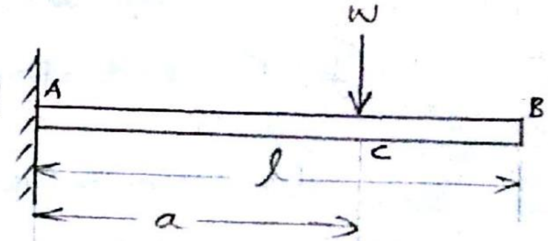
Here $\bar{x} = \frac{2}{3} l$.

$$\therefore y_{max} = \frac{\frac{1}{2} Wl^2 \times \frac{2}{3} l}{EI}$$

$$y_{max} = \frac{Wl^3}{3EI}$$

(2). Concentrated load at any point.

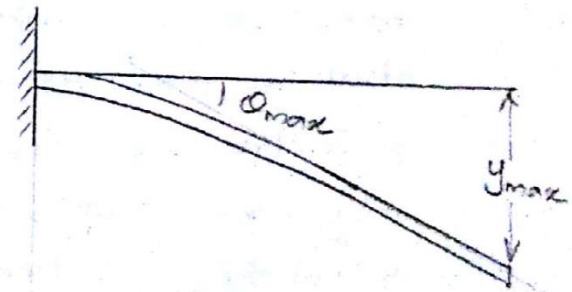
Fig: shows a cantilever with a concentrated load ~~from at~~ acting at c at a distance 'a' from fixed end.



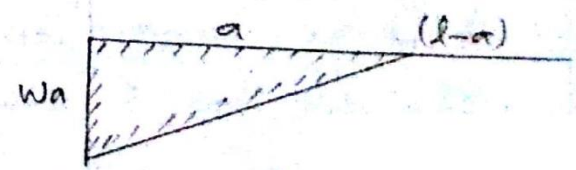
$$\theta_{max} = \frac{A}{EI}$$

Here, $A = \frac{1}{2} \times a \times wa$

$$= \frac{wa^2}{2}$$



$$\therefore \theta_{max} = \frac{wa^2}{2EI}$$



Now $y_{max} = \frac{A\bar{x}}{EI}$

Here, $\bar{x} = (l-a) + \frac{2}{3}a.$

$$\therefore y_{max} = \frac{\frac{1}{2}wa^2\left[(l-a) + \frac{2}{3}a\right]}{EI}$$

$$y_{max} = \frac{wa^3}{3EI} + \frac{wa^2}{2EI}(l-a)$$

At the point of application of the load, $y = \dfrac{\frac{1}{2}Wa^2 \times \frac{2a}{3}}{EI}$.

By transfering the ref line to the point of application of load.

$$y = \frac{Wa^3}{3EI}$$

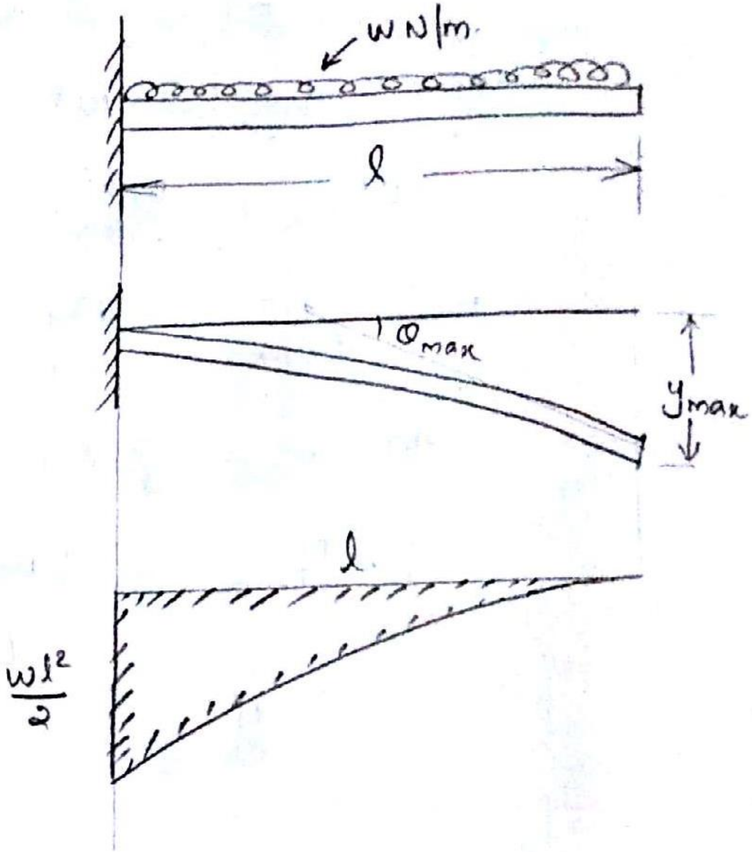
(3) Cantilever beam with UD load over entire span.

We have $\theta_{max} = \frac{A}{EI}$.

Here area of BM diagram $A = \frac{1}{3} bh$

$$= \frac{1}{3} \times l \times \frac{wl^2}{2}$$

$$\therefore \theta_{max} = \frac{wl^3}{6EI}.$$



$$y_{max} = \frac{A\bar{x}}{EI}.$$

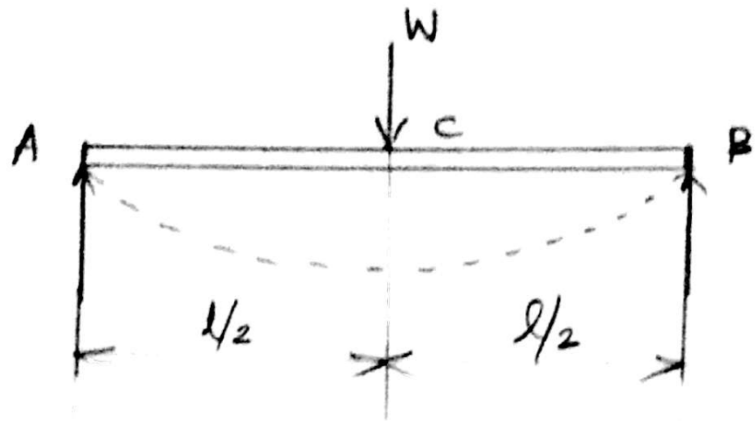
$$A = \frac{1}{3} \frac{wl^3}{2}.$$

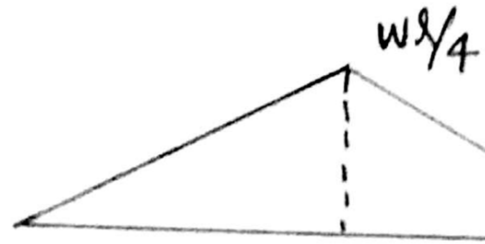
$$\bar{x} = \frac{3}{4} l.$$

$$\therefore y_{max} = \frac{wl^4}{8EI}$$

Case II : Simply Supported Beams.

1. Concentrated Load at the midspan





Area of the BMD between A and C $= \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{Wl}{4}$

$$= \frac{Wl^2}{16}$$

We have $\delta_{max} = \frac{A}{EI}$

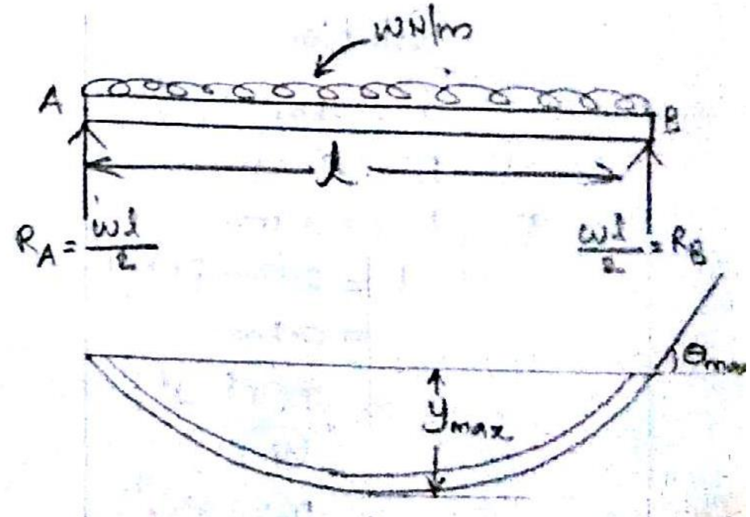
$\therefore$ Slope at A $= \frac{Wl^2}{16EI}$

and $y_{max} = \frac{A\bar{x}}{EI} = \frac{\left(\frac{1}{2} \cdot \frac{Wl}{4} \cdot \frac{l}{2}\right)\left(\frac{2}{3} \, l/2\right)}{EI}$

ie, $y_{max} = \frac{Wl^3}{48EI}$

(2) Uniformly distributed load.

In this case also the deflection will be max. at the midspan and slope will be max. at the ends.



Now $\theta_{max} = \frac{A}{EI}$.

Where $A = \frac{2}{3} \times \frac{l}{2} \times \frac{wl^2}{8}$.

$$\therefore \theta_{max} = \frac{wl^3}{24EI}$$

Now $y_{max} = \frac{A\bar{x}}{EI}$

$= \frac{A}{EI} \times \bar{x}$

$y_{max} = \frac{wl^3}{24EI} \times \frac{5}{8} \frac{l}{\times 2}$. Where $\bar{x} = \frac{5}{8} \frac{l}{2}$.

$$y_{max} = \frac{5wl^4}{384EI}$$

Conjugate Beam Method

We have, $EI \cdot \dfrac{d^2 y}{dx^2} = M$ or $\dfrac{d^2 y}{dx^2} = \dfrac{M}{EI}$

Differentiating it, $EI \cdot \dfrac{d^3 y}{dx^3} = \dfrac{dM}{dx} = F$

Differentiating it again, $EI \cdot \dfrac{d^4 y}{dx^4} = \dfrac{dF}{dx} = -w$

$$\frac{d^4 y}{dx^4} = -\frac{w}{EI} \quad \text{or} \quad \frac{d^2}{dx^2}\left(\frac{d^2 y}{dx^2}\right) = -\frac{w}{EI}$$

$$\frac{d^2}{dx^2}\left(\frac{M}{EI}\right) = -\frac{w}{EI} \quad \text{or} \quad \frac{d^2 M}{dx^2} = -w$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \qquad \text{(i)}$$

$$\frac{d^2M}{dx^2} = -w \qquad \text{(ii)}$$

- Thus as indicated by (ii), if *w* indicates the actual loading, and a bending moment diagram is drawn, it provides the bending moment at any cross-section of the beam.

- In a similar way it may be said from (i) that if the bending moment diagram (*M/EI*) is assumed as the loading diagram on the beam (the beam is known as *conjugate beam*) and a new bending moment diagram is constructed from this, the diagram will be a *deflection curve.*

A similar analogy for the slope can also be deduced

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

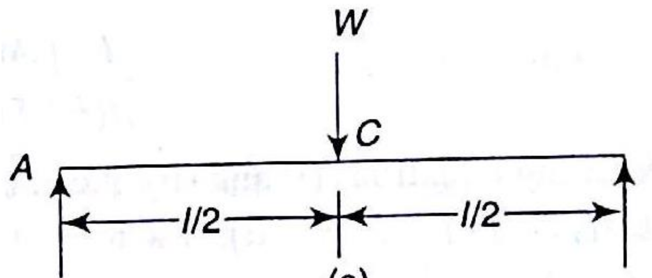
or $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI}$$

or $$\frac{d}{dx}(slope) = \frac{M}{EI} \qquad \text{——————} \qquad \text{(iii)}$$

Also, $$\frac{dF}{dx} = -w \qquad \text{——————} \qquad \text{(iv)}$$

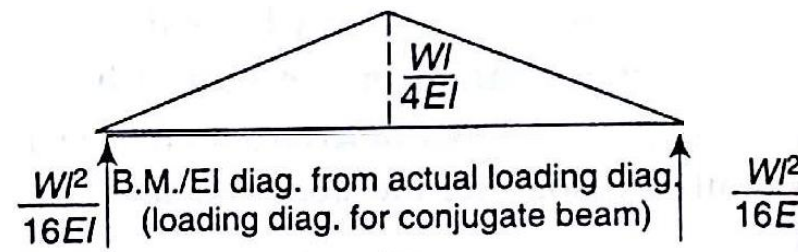
Thus shear force diagram drawn with *M/EI* as loading will provide the slope at any section.

Find expressions for the central deflection and the slope at the ends of a simply supported beam carrying a central load by conjugate beam method.



(a)

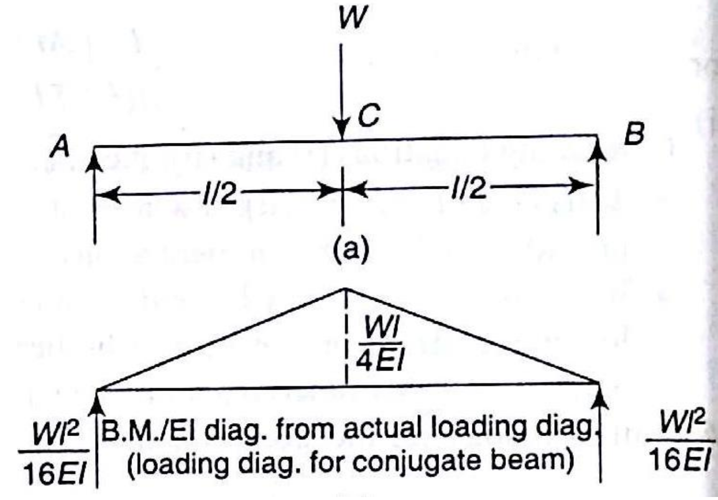
maximum bending moment at the centre is $Wl/4$,



(b)

Now, in the conjugate beam method, this diagram is to be considered as loading diagram first we need to find the reaction on the supports.

$$R_a = R_b = \frac{Wl}{4EI} \times \frac{l}{2} \times \frac{1}{2} = \frac{Wl^2}{16EI}$$



Deflections

Deflection y at any point at a distance x from A

= bending moment due to load on the conjugate beam

$$= \frac{Wl^2}{16EI}x - \frac{Wl/4EI}{l/2} \cdot x \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{Wl^2}{16EI}x - \frac{W}{12EI}x^3 = \frac{W}{48EI}(3l^2 x - 4x^3)$$

Maximum deflection at the centre $= \dfrac{W}{48EI}\left[3l^2 \cdot \dfrac{l}{2} - 4\left(\dfrac{l}{2}\right)^3\right] = \dfrac{Wl^3}{48EI}$

Slopes

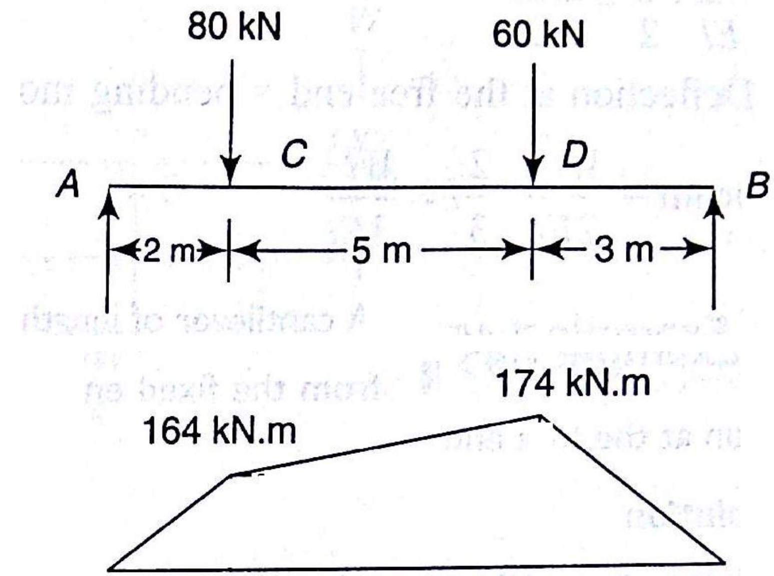
Slope at any point at a distance x from A

= Shearing force at the point due to load on the conjugate beam

$$= \frac{Wl^2}{16EI} - \frac{Wl/4EI}{l/2} \cdot x \cdot \frac{x}{2}$$

Slope at the ends $= \dfrac{Wl^2}{16EI}$ .....$(x = 0)$

A 10 m long simply supported beam *AB* carries loads of 80 kN and 60 kN at 2 m and 7 m respectively from A. $E = 200$ GPa and $I = 150 \times 10^6$ mm^4. Determine the deflection and slope under the loads using conjugate beam method.



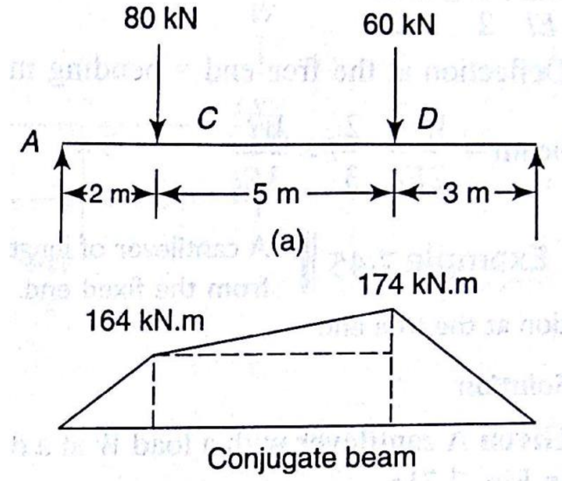
Taking moments about *A*,

$$10\,R_b = 80 \times 2 + 60 \times 7$$

or $$R_b = 58 \text{ kN}$$

$$R_a = 80 + 60 - 58 = 82 \text{ kN}$$

Bending moment at $C = 82 \times 2 = 164$ kN·m
Bending moment at $D = 58 \times 3 = 174$ kN·m



Conjugate beam

Bending moment (conjugate beam) diagram is shown in Fig. 7.69b.

Taking moments about B to find the reaction at A from conjugate loads,

$$10\,R_a = \left(164 \times 2 \times \frac{1}{2}\right)\left(\frac{2}{3} + 8\right) + 164 \times 5\left(3 + \frac{5}{2}\right) + (174 - 164) \times 5 \times \frac{1}{2}\left(3 + \frac{5}{3}\right) + 174 \times 3 \times \frac{1}{2} \times 2$$

$$10\,R_a = 1421.3 + 4510 + 116.7 + 522 \quad \text{or} \quad R_a = 657$$

$$R_b = 164 \times (2/2) + 164 \times 5 + (174 - 164) \times (5/2) + 174 \times (3/2) - 657 = 613$$

For conjugate beam

Shearing force at $C = 657 - 164 \times (2/2) = 493$

Shearing force at $D = -613 + 174 \times (3/2) = -352$

Bending moment at $C = 657 \times 2 - 164 \times (2/3) = 1204.7$

Bending moment at $D = 613 \times 3 - 174 \times (3/2) \times 1 = 1578$

Slope and deflection

$EI = 200 \times 10^6 \times (150 \times 10^{-6}) = 30\ 000\ \text{kN}\cdot\text{m}^2$

Slope at $C = 493/30\ 000 = 0.0164$ rad

Slope at $D = 352/30\ 000 = 0.0117$ rad

Deflection at $C = 1204.7/30\ 000 = 0.04016$ m $= 40.16$ mm

Deflection at $D = 1578/30\ 000 = 0.0526$ m $= 52.26$ mm